

SHARP LOCAL WELL-POSEDNESS OF KDV TYPE EQUATIONS WITH DISSIPATIVE PERTURBATIONS

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ABSTRACT. In this work, we study the initial value problems associated to some linear perturbations of KdV equations. Our focus is in the well-posedness issues for initial data given in the L^2 -based Sobolev spaces. We derive bilinear estimate in a space with weight in the time variable and obtain sharp local well-posedness results.

1. INTRODUCTION

In this article, continuing our earlier work [6], we consider the following initial value problems (IVPs)

$$\begin{aligned} v_t + v_{xxx} + \eta Lv + (v^2)_x &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ v(x, 0) &= v_0(x), \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} u_t + u_{xxx} + \eta Lu + (u_x)^2 &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= u_0(x), \end{aligned} \tag{1.2}$$

where $\eta > 0$ is a constant; $u = u(x, t)$, $v = v(x, t)$ are real valued functions and the linear operator L is defined via the Fourier transform by $\widehat{Lf}(\xi) = -\Phi(\xi)\hat{f}(\xi)$.

The Fourier symbol $\Phi(\xi)$ is of the form

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi), \tag{1.3}$$

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where $p \in \mathbb{R}^+$ and $|\Phi_1(\xi)| \leq C(1 + |\xi|^q)$ with $0 \leq q < p$. We note that the symbol $\Phi(\xi)$ is a real valued function which is bounded above; i.e., there is a constant C such that $\Phi(\xi) < C$ (see Lemma 2.2 below). In our earlier work [6], we considered a particular case of $\Phi(\xi)$ in the following form

$$\tilde{\Phi}(\xi) = \sum_{j=0}^n \sum_{i=0}^{2m} c_{i,j} \xi^i |\xi|^j, \quad c_{i,j} \in \mathbb{R}, \quad c_{2m,n} = -1, \quad (1.4)$$

with $p := 2m + n$.

We observe that, if u is a solution of (1.2) then $v = u_x$ is a solution of (1.1) with initial data $v_0 = (u_0)_x$. That is why (1.1) is called the derivative equation of (1.2).

In this work, we are interested in investigating the well-posedness results to the IVPs (1.2) and (1.1) for given data in the low regularity Sobolev spaces $H^s(\mathbb{R})$. Recall that, for $s \in \mathbb{R}$, the L^2 -based Sobolev spaces $H^s(\mathbb{R})$ are defined by

$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

and $\hat{f}(\xi)$ is the usual Fourier transform given by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

The factor $\frac{1}{\sqrt{2\pi}}$ in the definition of the Fourier transform does not alter our analysis, so we will omit it.

The notion of well-posedness we use is the standard one. We say that an IVP for given data in a Banach space X is locally well-posed, if there exists a certain time interval $[0, T]$ and a unique solution depending continuously upon the initial data and the solution satisfies the persistence property; i.e., the solution describes a continuous curve in X in the time interval $[0, T]$. If the above properties are true for any time interval, we say that the IVP is globally well-posed.

With motivation from the work in [12], we introduce function spaces that will be used to prove the local well-posedness results. For $p > 0$ and $t \in [0, T]$ with $0 \leq T \leq 1$, these spaces are defined with weight in time variable via the norms

$$\|f\|_{X_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{|s|}{p}} \|f(t)\|_{L^2} \right\}, \quad (1.5)$$

and

$$\|f\|_{Y_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{1+|s|}{p}} \|\partial_x f(t)\|_{L^2} \right\}, \quad (1.6)$$

and will be used to prove local well-posedness for the IVPs (1.1) and (1.2) respectively.

We use notation $\langle \cdot \rangle = (1 + |\cdot|)$.

Now, we state the main results of this work. The first result deals with the local well-posedness results for the IVP (1.1).

Theorem 1.1. *Let $\eta > 0$ be fixed and $\Phi(\xi)$ be as given by (1.3) with $p > 3$ as the order of the leading term, then the IVP (1.1) is locally well-posed for any data $v_0 \in H^s(\mathbb{R})$, whenever $s > -\frac{p}{2}$. Moreover, the map $v_0 \mapsto v(t)$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap X_T^s$*

The the second result deals the same for the IVP (1.2), with low regularity data.

Theorem 1.2. *Let $\eta > 0$ be fixed and $\Phi(\xi)$ be as given by (1.3) with $p > 3$ as the order of the leading term, then the IVP (1.2) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, whenever $s > 1 - \frac{p}{2}$. Moreover, the map $u_0 \mapsto u(t)$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap Y_T^s$*

The third and the fourth results deal with the ill-posedness issues for the IVP (1.1) and (1.2) respectively and show that the results obtained in Theorems 1.1 and 1.2 are sharp.

Theorem 1.3. *Let $s < -\frac{p}{2}$, then there does not exist any $T > 0$ such that the IVP (1.1) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map*

$$v_0 \mapsto v(t), \quad t \in [0, T], \quad (1.7)$$

is C^2 -differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

Theorem 1.4. *Let $s < 1 - \frac{p}{2}$, then there does not exist any $T > 0$ such that the IVP (1.2) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map*

$$u_0 \mapsto u(t), \quad t \in [0, T], \quad (1.8)$$

is C^2 -differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

Remark 1.5. *As it can be seen in the proofs below, our method in this article holds only for $-\frac{p}{2} < s < \frac{p}{2}$ in Theorem 1.1 and for $1 - \frac{p}{2} < s < 1 + \frac{p}{2}$ in Theorem 1.2 considering $p > 3$. However, for $s \geq \frac{p}{2}$ (Theorem 1.1) and $s \geq 1 + \frac{p}{2}$ (Theorem 1.2) we already have proved local well-posedness in our earlier work [6]. As far as we know, the case when $2 < p \leq 3$ is an open problem and is a subject of research in our ongoing project. For motivation, we refer to the recent work of Molinet and Rebaud [17] where the authors proved sharp local well-posedness for $s > -1$ considering $p = 2$ in the frame work of the Fourier transform restriction norm spaces introduced by Bourgain [5].*

Remark 1.6. *We believe that this method can be extended to address the local well-posedness issues for the IVPs (1.1) and (1.2) with generalized nonlinearity, (i.e., for $\partial_x(v^{k+1})$ and $(\partial_x u)^{k+1}$, $k > 1$, respectively). In this case the spaces X_T^s and Y_T^s should be adapted according as the values of k and p . A work in this direction is in progress.*

In what follows, we present some particular examples that belong to the class considered in (1.1) and (1.2) and discuss the known well-posedness results about them.

The first examples belonging to the classes (1.1) and (1.2) are

$$\begin{aligned} v_t + v_{xxx} - \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) + (v^2)_x &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ v(x, 0) &= v_0(x), \end{aligned} \quad (1.9)$$

and

$$\begin{aligned} u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + (u_x)^2 &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= u_0(x), \end{aligned} \quad (1.10)$$

respectively, where \mathcal{H} denotes the Hilbert transform

$$\mathcal{H}g(x) = \text{P. V.} \frac{1}{\pi} \int \frac{g(x - \xi)}{\xi} d\xi;$$

$u = u(x, t)$, $v = v(x, t)$ are real-valued functions and $\eta > 0$ is a constant.

The equation in (1.9) was derived by Ostrovsky et al [18] to describe the radiational instability of long waves in a stratified shear flow. Recently, Carvajal and Scialom [8] considered the IVP (1.9) and proved the local well-posedness results for given data in H^s , $s \geq 0$. They also obtained an *a priori* estimate for given data in $L^2(\mathbb{R})$ there by proving global well-posedness result. The earlier well-posedness results for (1.9) can be found in [1], where for given data in $H^s(\mathbb{R})$, local well-posedness when $s > 1/2$ and global well-posedness when $s \geq 1$ have been proved. In [1], IVP (1.10) is also considered to prove global well-posedness for given data in $H^s(\mathbb{R})$, $s \geq 1$. These results are further improved in our recent work [7] and [6], where we proved that the IVPs (1.9) and (1.10) for given data in $H^s(\mathbb{R})$ are locally well-posed whenever $s > -\frac{3}{4}$ and $s > \frac{1}{4}$ respectively. To obtain these results we followed the techniques developed by Bourgain [5] and Kenig, Ponce and Vega [15] (see also [20]). The main ingredients in the proof are estimates in the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$. The main idea in [7] and [6] is to use the usual Bourgain space associated to the KdV equation instead of that associated to the linear part of the IVPs (1.1) and (1.2). For the well-posedness issues in the periodic setting we refer to [9].

Another two models that fit in the classes (1.2) and (1.1) respectively are the Korteweg-de Vries-Kuramoto Sivashinsky (KdV-KS) equation

$$\begin{aligned} u_t + u_{xxx} + \eta(u_{xx} + u_{xxx}) + (u_x)^2 &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= u_0(x), \end{aligned} \tag{1.11}$$

and its derivative equation

$$\begin{aligned} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + vv_x &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ v(x, 0) &= v_0(x), \end{aligned} \tag{1.12}$$

where $u = u(x, t)$, $v = v(x, t)$ are real-valued functions and $\eta > 0$ is a constant.

The KdV-KS equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (see [11, 21]). The KdV-KS equation is very interesting in the sense that it combines the dispersive characteristics of the Korteweg-de Vries equation and dissipative characteristics of the Kuramoto-Sivashinsky equation. Also, it is worth noticing that (1.12) is a particular case of the Benney-Lin equation [2, 21]; i.e.,

$$\begin{aligned} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + \beta v_{xxxxx} + vv_x &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ v(x, 0) &= v_0(x), \end{aligned} \tag{1.13}$$

when $\beta = 0$.

The IVPs (1.11) and (1.12) were studied by Biagioni, Bona, Iorio and Scialom [3]. The authors in [3] proved that the IVPs (1.11) and (1.12) are locally well-posed for given data in H^s , $s \geq 1$ with $\eta > 0$. They also constructed appropriate *a priori* estimates and used them to prove global well-posedness too. The limiting behavior of solutions as the dissipation tends to zero (i.e., $\eta \rightarrow 0$) has also been studied in [3]. The IVP (1.13) associated to the Benney-Lin equation is also widely studied in the literature [2, 4, 21]. Regarding well-posedness issues for the IVP (1.13) the work of Biagioni and Linares [4] is worth mentioning, where they proved global well-posedness for given data in $L^2(\mathbb{R})$. For the sharp well-posedness result for the KdV-KS equation we refer to the recent work of Pilod in [19] where the author proved local well-posedness in $H^s(\mathbb{R})$ for $s > -1$ and ill-posedness for $s < -1$.

Now we consider the IVP associated to the linear parts of (1.1) and (1.2),

$$\begin{aligned} w_t + w_{xxx} + \eta Lw &= 0, \quad x, t \geq 0, \\ w(0) &= w_0. \end{aligned} \tag{1.14}$$

The solution to (1.14) is given by $w(x, t) = V(t)w_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)w_0}(\xi) = e^{it\xi^3 + \eta t\Phi(\xi)} \widehat{w_0}(\xi). \tag{1.15}$$

This paper is organized as follows: In Section 2, we prove some preliminary estimates. Sections 3 and 4 are dedicated to prove the local well-posedness and ill-posedness results respectively.

2. PRELIMINARY ESTIMATES

This section is devoted to obtain linear and nonlinear estimates that are essential in the proof of the main results. We start with following estimate that the Fourier symbol defined in (1.3) satisfies.

Lemma 2.1. *There exists $M > 0$ large such that for all $|\xi| \geq M$, one has that*

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi) < -1, \tag{2.1}$$

$$\frac{\Phi_1(\xi)}{|\xi|^p} \leq \frac{1}{2}, \tag{2.2}$$

and

$$|\Phi(\xi)| \geq \frac{|\xi|^p}{2}. \tag{2.3}$$

Proof. The inequalities (2.1) and (2.2) are direct consequences of

$$\lim_{\xi \rightarrow \infty} \frac{\Phi_1(\xi) + 1}{|\xi|^p} = 0 \quad \text{and} \quad \lim_{\xi \rightarrow \infty} \frac{\Phi_1(\xi)}{|\xi|^p} = 0,$$

respectively.

The estimate (2.3) follows from (2.1) and (2.2). In fact, for $|\xi| > M$

$$|\Phi(\xi)| = |\xi|^p - \Phi_1(\xi) \geq \frac{|\xi|^p}{2}, \tag{2.4}$$

and this concludes the proof of the (2.3). \square

Lemma 2.2. *The Fourier symbol $\Phi(\xi)$ given by (1.3) is bounded from above and the following estimate holds true*

$$\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M}. \quad (2.5)$$

Proof. From Lemma 2.1, there is $M > 1$ large enough such that for $|\xi| \geq M$ one has $\Phi(\xi) < -1$. Consequently, $e^{t\Phi(\xi)} \leq e^{-t} \leq 1$. Now for $|\xi| < M$, it is easy to get $\Phi(\xi) < C_M$, so that $e^{t\Phi(\xi)} \leq e^{TC_M}$. Therefore, in any case

$$\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M}.$$

\square

The following result is an elementary fact from calculus.

Lemma 2.3. *Let $f(t) = t^a e^{tb}$ with $a > 0$ and $b < 0$, then for all $t \geq 0$ one has*

$$f(t) \leq \left(\frac{a}{|b|}\right)^a e^{-a}. \quad (2.6)$$

Lemma 2.4. *Let $0 < T \leq 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have*

$$\|V(t)u_0\|_{X_T^s} \lesssim e^{C_M T} \|u_0\|_{H^s}, \quad (2.7)$$

where the constant C_M depends on M with M as in Lemma 2.1.

Proof. We start by estimating the first component of the X_T^s -norm. We have that

$$\|V(t)u_0\|_{H^s} = \|\langle \xi \rangle^s e^{t\Phi(\xi)} \widehat{u}_0(\xi)\|_{L^2} \leq \|e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \quad (2.8)$$

Using (2.5) in (2.8), we get

$$\|V(t)u_0\|_{H^s} \leq e^{TC_M} \|u_0\|_{H^s}. \quad (2.9)$$

Now, we move to estimate the second component of the X_T^s -norm. The case $s \geq 0$ is quite easy, so we consider only the case when $s < 0$. Using Plancherel, we have

$$\begin{aligned} t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} &= t^{\frac{|s|}{p}} \|e^{t\Phi(\xi)} \widehat{u_0}\|_{L^2} \\ &= t^{\frac{|s|}{p}} \|\langle \xi \rangle^{-s} e^{t\Phi(\xi)} \langle \xi \rangle^s \widehat{u_0}\|_{L^2} \\ &\leq t^{\frac{|s|}{p}} \|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \end{aligned} \quad (2.10)$$

Since $\langle \xi \rangle^{|s|} \lesssim 1 + |\xi|^{|s|}$, from (2.10), one obtains

$$t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} \leq t^{\frac{|s|}{p}} \left[\|e^{t\Phi(\xi)}\|_{L^\infty} + \| |\xi|^{|s|} e^{t\Phi(\xi)} \|_{L^\infty} \right] \|u_0\|_{H^s}. \quad (2.11)$$

From (2.5), we have $\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M}$. To estimate $\| |\xi|^{|s|} e^{t\Phi(\xi)} \|_{L^\infty}$ we proceed as follows.

$$\| |\xi|^{|s|} e^{t\Phi(\xi)} \|_{L^\infty} \leq \| |\xi|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}} \|_{L^\infty} + \| |\xi|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}} \|_{L^\infty}. \quad (2.12)$$

For the low-frequency part, it is easy to get

$$\| |\xi|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}} \|_{L^\infty} \leq M^{|s|} e^{C_M T}. \quad (2.13)$$

Now, we move to estimate the high-frequency part $\| |\xi|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}} \|_{L^\infty}$ in (2.12). For this, we make use of the time weight in the definition of X_T^s -norm and define for $|\xi| > M$, $g(t, \xi) := t^{\frac{|s|}{p}} |\xi|^{|s|} e^{t\Phi(\xi)}$. Using the estimate (2.6) from Lemma 2.3, we get

$$g(t, \xi) \leq \left(\frac{|s|}{p|\Phi(\xi)|} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}} |\xi|^{|s|}. \quad (2.14)$$

Since $M > 1$ is large, an application of the estimate (2.3) from Lemma 2.1 in (2.14), yields

$$g(t, \xi) \leq \left(\frac{2|s|}{p|\xi|^p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}} |\xi|^{|s|} \leq \left(\frac{2|s|}{p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}}. \quad (2.15)$$

In light of the estimate (2.15), one obtains that

$$t^{\frac{|s|}{p}} \| |\xi|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}} \|_{L^\infty} \leq \left(\frac{2|s|}{p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}}. \quad (2.16)$$

Inserting estimates (2.5), (2.13) and (2.16) in (2.11), we get

$$t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} \leq \left(\frac{2|s|}{p}\right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}} \|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}. \quad (2.17)$$

The conclusion of the Lemma follows from (2.9) and (2.17). \square

Lemma 2.5. *Let $0 < T \leq 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have*

$$\|V(t)u_0\|_{Y_T^s} \lesssim e^{TC_M} \|u_0\|_{H^s}, \quad (2.18)$$

where the constant C_M depends on M with M as in Lemma 2.1.

Proof. The estimate for the first component of the Y_T^s -norm has already been obtained in (2.9). In what follows, we estimate the second component of the Y_T^s -norm. We only consider the case when $s < 0$. In the case when $s \geq 0$ the estimates follow easily. Using Plancherel, we have

$$\begin{aligned} t^{\frac{1+|s|}{p}} \|\partial_x V(t)u_0\|_{L^2} &= t^{\frac{1+|s|}{p}} \|\xi e^{t\Phi(\xi)} \widehat{u}_0\|_{L^2} \\ &= t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{-s} e^{t\Phi(\xi)} \langle \xi \rangle^s \widehat{u}_0\|_{L^2} \\ &\leq t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \end{aligned} \quad (2.19)$$

Now,

$$\begin{aligned} t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} &\leq t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}}\|_{L^\infty} + t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}}\|_{L^\infty} \\ &=: J_1 + J_2. \end{aligned} \quad (2.20)$$

Since $\langle \xi \rangle^{|s|} \lesssim 1 + |\xi|^{|s|}$, and $t \in [0, T]$ with $0 \leq T \leq 1$, we have

$$J_1 \lesssim C_M t^{\frac{1+|s|}{p}} \leq C_M. \quad (2.21)$$

Now, we move to estimate the high-frequency part J_2 . For this, we use the estimate (2.6) from Lemma 2.3 with $b = \Phi(\xi) < 0$ and $a = \frac{1+|s|}{p}$, we get

$$e^{t\Phi(\xi)} \leq \left(\frac{ae^{-1}}{|\Phi(\xi)|} \right)^a \frac{1}{t^a}. \quad (2.22)$$

Since $M > 1$ is large, $\langle \xi \rangle^{|s|} \lesssim |\xi|^{|s|}$, an application of the estimate (2.3) from Lemma 2.1 in (2.22), yields

$$t^{\frac{1+|s|}{p}} |\xi|^{1+|s|} \left(\frac{ae^{-1}}{|\Phi(\xi)|} \right)^a \frac{1}{t^a} \lesssim t^{\frac{1+|s|}{p}-a} |\xi|^{1+|s|-ap} \lesssim C_M, \quad (2.23)$$

and consequently

$$J_2 \lesssim C_M. \quad (2.24)$$

The conclusion of the Lemma follows from (2.9), (2.19), (2.20), (2.21) and (2.24). \square

Lemma 2.6. *Let $-\frac{p}{2} < s$, $p > 3$, $0 < T \leq 1$ and $\tau \in [0, T]$. Then we have*

$$\|\xi \langle \xi \rangle^s e^{\tau \Phi(\xi)}\|_{L_\xi^2} \lesssim \frac{1}{\tau^{\frac{1}{2} + \frac{s}{p}}}, \quad (2.25)$$

and

$$\|\xi e^{\tau \Phi(\xi)}\|_{L_\xi^2} \lesssim \frac{1}{\tau^{\frac{3+}{2p}}}. \quad (2.26)$$

Proof. In order to prove (2.25), let M be as in Lemma 2.1, we decompose the integral

$$\|\xi \langle \xi \rangle^s e^{\tau \Phi(\xi)}\|_{L_\xi^2}^2 = \int_{|\xi| \leq M} \xi^2 \langle \xi \rangle^{2s} e^{2\tau \Phi(\xi)} d\xi + \int_{|\xi| \geq M} \xi^2 \langle \xi \rangle^{2s} e^{2\tau \Phi(\xi)} d\xi =: I_1 + I_2. \quad (2.27)$$

In the first integral, since $1 + 2\frac{s}{p} > 0$ and $\tau \in [0, 1]$ we have

$$I_1 \leq \int_{|\xi| \leq M} M^2 e^{2C\tau} d\xi \leq 2M^3 e^{2C\tau} \leq \frac{2M^3 e^{2C}}{\tau^{1-2\frac{|s|}{p}}}. \quad (2.28)$$

Now, we consider the second integral in (2.27). For sufficiently large M , if we take $b = 2\Phi(\xi) < 0$ (see Lemma 2.1) and $a = 1 + 2\frac{s}{p} > 0$, then using the estimates (2.6) and (2.3), we get

$$I_2 \lesssim \frac{1}{\tau^a} \int_{|\xi| \geq M} \xi^2 \langle \xi \rangle^{2s} \frac{1}{|\Phi(\xi)|^a} d\xi \leq \frac{1}{\tau^{1+2\frac{s}{p}}} \int_{|\xi| \geq M} \frac{1}{|\xi|^{-2-2s+p(1+2\frac{s}{p})}} d\xi \lesssim \frac{1}{\tau^{1+2\frac{s}{p}}},$$

where in the last inequality the fact that $-2 - 2s + p(1 + 2s/p) = p - 2 > 1$ has been used, and this proves (2.25).

The proof of the (2.26) is very similar. Again we consider M as in Lemma 2.1, and decompose the integral

$$\|\xi e^{\tau\Phi(\xi)}\|_{L_\xi^2}^2 = \int_{|\xi| \leq M} \xi^2 e^{2\tau\Phi(\xi)} d\xi + \int_{|\xi| \geq M} \xi^2 e^{2\tau\Phi(\xi)} d\xi =: J_1 + J_2. \quad (2.29)$$

Since $\frac{p}{3} > 0$ and $\tau \in [0, 1]$, we have

$$J_1 \leq \int_{|\xi| \leq M} M^2 e^{2C\tau} d\xi \leq 2M^3 e^{2C\tau} \leq \frac{2M^3 e^{2C}}{\tau^{\frac{3^+}{p}}}. \quad (2.30)$$

Similarly as in the case of I_2 , using (2.6) with $b = 2\Phi(\xi) < 0$ and $a = \frac{3^+}{p} > 0$, and estimate (2.3), we obtain

$$J_2 \lesssim \frac{1}{\tau^a} \int_{|\xi| \geq M} \xi^2 \frac{1}{|\Phi(\xi)|^a} d\xi \leq \frac{1}{\tau^{\frac{3^+}{p}}} \int_{|\xi| \geq M} \frac{1}{|\xi|^{-2+p(\frac{3^+}{p})}} d\xi \lesssim \frac{1}{\tau^{\frac{3^+}{p}}},$$

where in the last inequality the fact that $-2 + p(\frac{3^+}{p}) > 1$, has been used, and this proves (2.26). \square

Proposition 2.7. *Let $-\frac{p}{2} < s < \frac{p}{2}$, $p > 3$, $0 < T \leq 1$ and $t \in [0, T]$. Then we have*

$$\left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{X_T^s} \leq T^\alpha \|u\|_{X_T^s} \|v\|_{X_T^s}, \quad (2.31)$$

where $\alpha = \frac{2s+p}{2p} > 0$.

Proof. Using the definition of $V(t)$ and Minkowski's inequality, we have

$$\begin{aligned} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} &\leq \int_0^t \|\xi \langle \xi \rangle^s e^{(t-t')\Phi(\xi)} (\widehat{u(t')} * \widehat{v(t')}) dt'\|_{L_\xi^2} \\ &\leq \int_0^t \|\xi \langle \xi \rangle^s e^{(t-t')\Phi(\xi)}\|_{L_\xi^2} \|(\widehat{u(t')} * \widehat{v(t')})(\xi)\|_{L_\xi^\infty} dt'. \end{aligned} \quad (2.32)$$

The Young's inequality, Plancherel identity and definition of X_T^s norm yield

$$\|(\widehat{u(t')} * \widehat{v(t')})(\xi)\|_{L_\xi^\infty} \leq t'^{-\frac{2|s|}{p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \quad (2.33)$$

Combining inequalities (2.32), (2.33) and inequality (2.25) in Lemma 2.6, we get

$$\left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^t \frac{1}{|t-t'|^{\frac{1}{2}+\frac{s}{p}} |t'|^{\frac{2|s|}{p}}} dt' \quad (2.34)$$

Making a change of variables $t' = t\tau$, we get

$$\begin{aligned} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} &\lesssim t^{\frac{p+2s}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{1}{2}+\frac{s}{p}} |\tau|^{\frac{2|s|}{p}}} d\tau \\ &\lesssim t^{\frac{p+2s}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \end{aligned} \quad (2.35)$$

Similarly inequality (2.26) in Lemma 2.6 and (2.33) give

$$t^{\frac{|s|}{p}} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{L^2} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s} t^{\frac{|s|}{p}} \int_0^t \frac{1}{|t-t'|^{\frac{3+}{2p}} |t'|^{\frac{2|s|}{p}}} d\tau \quad (2.36)$$

Again, Making a change of variables $t' = t\tau$, one has

$$\begin{aligned} t^{\frac{|s|}{p}} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{L^2} &\lesssim t^{\frac{2p+2s-3^+}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{3+}{2p}} |\tau|^{\frac{2|s|}{p}}} d\tau \\ &\lesssim t^{\frac{2p+2s-3^+}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \end{aligned} \quad (2.37)$$

□

Proposition 2.8. *Let $1 - \frac{p}{2} < s < \frac{p}{2} - 1$, $p > 3$, $0 < T \leq 1$ and $t \in [0, T]$. Then we have*

$$\left\| \int_0^t V(t-t') (u_x v_x)(t') dt' \right\|_{Y_T^s} \lesssim T^\theta \|u\|_{Y_T^s} \|v\|_{Y_T^s}, \quad (2.38)$$

where $\theta = \frac{p-2+2s}{2p} > 0$.

Proof. We start considering the H^s part of the Y_T^s -norm. Using the definition of $V(t)$ and Minkowski's inequality, we have

$$\begin{aligned} \left\| \int_0^t V(t-t') (u_x v_x)(t') dt' \right\|_{H^s} &\leq \int_0^t \left\| \langle \xi \rangle^s e^{(t-t')\Phi(\xi)} (\widehat{u_x(t')} * \widehat{v_x(t')}) dt' \right\|_{L_\xi^2} \\ &\leq \int_0^t \left\| \langle \xi \rangle^s e^{(t-t')\Phi(\xi)} \right\|_{L_\xi^2} \left\| (\widehat{u_x(t')} * \widehat{v_x(t')})(\xi) \right\|_{L_\xi^\infty} dt'. \end{aligned} \quad (2.39)$$

The Young's inequality, Plancherel identity and definition of Y_T^s norm yield

$$\|(\widehat{u_x(t')} * \widehat{v_x(t')})(\xi)\|_{L_\xi^\infty} \leq t'^{-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \quad (2.40)$$

For M large as in Lemma 2.1, we have

$$\|\langle \xi \rangle^s e^{\tau \Phi(\xi)}\|_{L^2}^2 = \int_{|\xi| \leq M} \langle \xi \rangle^{2s} e^{2\tau \Phi(\xi)} d\xi + \int_{|\xi| > M} \langle \xi \rangle^{2s} e^{2\tau \Phi(\xi)} d\xi =: A + B. \quad (2.41)$$

Now, for $\tau \in [0, T]$ and $a = \frac{p-2+2s}{p} > 0$, one has

$$A \leq C_M e^{TC_M} \lesssim \frac{1}{\tau^a}. \quad (2.42)$$

To obtain estimate for the high frequency part B , we use estimate (2.6) with $a = \frac{p-2+2s}{p} > 0$ and $b = 2\Phi(\xi) < 0$, to obtain

$$B \leq \int_{|\xi| > M} \frac{|\xi|^{2s}}{\tau^a} \frac{(ae^{-1})^a}{|\Phi(\xi)|^a} d\xi \lesssim \int_{|\xi| > M} \frac{1}{|\xi|^{pa-2s}} \frac{1}{\tau^a} d\xi \lesssim \frac{1}{\tau^a}, \quad (2.43)$$

where in the last inequality $pa - 2s > 1$ has been used.

Inserting (2.40), (2.42) and (2.43) in (2.39), we get

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^t \frac{1}{|t-t'|^{\frac{a}{2}} |t'|^{\frac{2(1+|s|)}{p}}} dt'. \quad (2.44)$$

Making a change of variables $t' = t\tau$, one obtains

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim t^{1-\frac{a}{2}-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{a}{2}} |\tau|^{\frac{2(1+|s|)}{p}}} d\tau. \quad (2.45)$$

For our choice of $a = \frac{p-2+2s}{p}$ and $s > 1 - \frac{p}{2}$ the integral in the RHS of (2.45) is finite, so we deduce that

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim t^{\frac{p-2+2s}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \quad (2.46)$$

Now, we move to estimate the second part of the Y_T^s -norm.

$$\begin{aligned} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} &\leq \int_0^t \|\xi e^{(t-t')\Phi(\xi)} \widehat{u_x(t')} * \widehat{v_x(t')}\|_{L^2} dt' \\ &\leq \int_0^t \|\widehat{u_x(t')} * \widehat{v_x(t')}\|_{L^\infty} \|\xi e^{(t-t')\Phi(\xi)}\|_{L^2} dt'. \end{aligned} \quad (2.47)$$

We have that $\|\widehat{u_x(t)} * \widehat{v_x(t)}\|_{L^\infty} \leq t^{-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}$. Taking $a = \frac{3^+}{2p}$, from (2.26), one gets $\|\xi e^{\tau\Phi(\xi)}\|_{L^2} \lesssim \frac{1}{\tau^a}$. So, from (2.47), one can deduce

$$\left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} \lesssim \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^t \frac{1}{|t-t'|^a |t'|^{\frac{2(1+|s|)}{p}}} dt'. \quad (2.48)$$

Making a change of variables $t' = t\tau$, one obtains from (2.48)

$$t^{\frac{(1+|s|)}{p}} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} \lesssim t^{1-\frac{(1+|s|)}{p}-a} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^1 \frac{1}{|1-\tau|^a |\tau|^{\frac{2(1+|s|)}{p}}} d\tau. \quad (2.49)$$

For our choice of $a = \frac{3^+}{2p}$ and $s > 1 - \frac{p}{2}$ the integral in the RHS of (2.49) is finite. Therefore, from (2.49), we obtain

$$\begin{aligned} t^{\frac{1+|s|}{p}} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} &\lesssim t^{\frac{2p+2s-5^+}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \\ &\lesssim t^{\frac{p-2+2s}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \end{aligned} \quad (2.50)$$

Combining (2.46) and (2.50) we get the required estimate (2.38). \square

3. PROOF OF THE WELL-POSEDNESS RESULT

This section is devoted to provide proofs of the local well-posedness results stated in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Now consider the IVP (1.1) in its equivalent integral form

$$v(t) = V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t') dt', \quad (3.1)$$

where $V(t)$ is the semigroup associated with the linear part given by (1.15).

We define an application

$$\Psi(v)(t) = V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t') dt'. \quad (3.2)$$

For $-\frac{p}{2} \leq s \leq \frac{p}{2}$, $r > 0$ and $0 < T \leq 1$, let us define a ball

$$B_r^T = \{f \in X_T^s; \|f\|_{X_T^s} \leq r\}.$$

We will prove that there exists $r > 0$ and $0 < T \leq 1$ such that the application Ψ maps B_r^T into B_r^T and is a contraction. Let $v \in B_r^T$. By using Lemma 2.4 and Proposition 2.7, we get

$$\|\Psi(v)\|_{X_T^s} \leq c\|v_0\|_{H^s} + cT^\alpha \|v\|_{X_T^s}^2, \quad (3.3)$$

where $\alpha = \frac{2s+p}{2p} > 0$.

Now, using the definition of B_r^T , one obtains

$$\|\Psi(v)\|_{X_T^s} \leq \frac{r}{4} + cT^\alpha r^2 \leq \frac{r}{2}, \quad (3.4)$$

where we have chosen $r = 4c\|v_0\|_{H^s}$ and $cT^\alpha r^2 = 1/4$. Therefore, from (3.4) we see that the application Ψ maps B_r^T into itself. A similar argument proves that Ψ is a contraction. Hence Ψ has a fixed point v which is a solution of the IVP (1.1) such that $v \in C([0, T], H^s(\mathbb{R}))$. \square

Proof of Theorem 1.2. The proof of this theorem is similar to the one presented for Theorem 1.1. Here, we will use the estimates from Lemma 2.5 and Proposition 2.8. So, we omit the details. \square

4. ILL-POSEDNESS RESULT

In this section we will use the ideas presented in [17] to prove the ill-posedness result stated in Theorem 1.3 and 1.4. The idea is to prove that there are no spaces X_T^s and Y_t^s that are continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ on which a contraction mapping argument can be applied. We start with the following result.

Proposition 4.1. *Let $s < -\frac{p}{2}$ and $T > 0$. Then there does not exist a space X_T^s continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that*

$$\|V(t)v_0\|_{X_T^s} \lesssim \|v_0\|_{H^s}, \quad (4.1)$$

$$\left\| \int_0^t V(t-t') \partial_x(v(t'))^2 dt' \right\|_{X_T^s} \lesssim \|v\|_{X_T^s}^2. \quad (4.2)$$

Proof. The proof follows a contradiction argument. If possible, suppose that there exists a space X_T^s that is continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that the estimates (4.1) and (4.2) hold true. If we consider $v = V(t)v_0$ then from (4.1) and (4.2), we get

$$\left\| \int_0^t V(t-t') \partial_x [V(t')v_0]^2 dt' \right\|_{H^s} \lesssim \|v_0\|_{H^s}^2. \quad (4.3)$$

The main idea to complete the proof is to find an appropriate initial data v_0 for which the estimate (4.3) fails to hold whenever $s < -\frac{p}{2}$.

Let $N \gg 1$, $0 < \gamma \ll 1$, $I_N := [N, N + 2\gamma]$ and define an initial data via Fourier Transform

$$\widehat{v}_0(\xi) := N^{-s} \gamma^{-\frac{1}{2}} [\chi_{\{I_N\}}(\xi) + \chi_{\{-I_N\}}(\xi)]. \quad (4.4)$$

A simple calculation shows that $\|v_0\|_{H^s} \sim 1$.

Now, we move to calculate the H^s norm of $f(x, t)$, where

$$f(x, t) := \int_0^t V(t-t') \partial_x [V(t')v_0]^2 dt'. \quad (4.5)$$

Taking the Fourier transform in the space variable x , we get

$$\begin{aligned} \widehat{f(t)}(\xi) &= \int_0^t e^{i(t-t')\xi^3 + (t-t')\Phi(\xi)} i\xi \left(\widehat{V(t')v_0} * \widehat{V(t')v_0} \right) (\xi) dt' \\ &= \int_0^t e^{i(t-t')\xi^3 + (t-t')\Phi(\xi)} i\xi \int_{\mathbb{R}} \widehat{v}_0(\xi - \xi_1) \widehat{v}_0(\xi_1) e^{it'\xi_1^3 + t'\Phi(\xi_1) + it'(\xi - \xi_1)^3 + t'\Phi(\xi - \xi_1)} d\xi_1 dt' \\ &= i\xi e^{it\xi^3 + t\Phi(\xi)} \int_{\mathbb{R}} \widehat{v}_0(\xi - \xi_1) \widehat{v}_0(\xi_1) \int_0^t e^{it'[-\xi^3 + \xi_1^3 + (\xi - \xi_1)^3] + t'[\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)]} dt' d\xi_1. \end{aligned} \quad (4.6)$$

We have that

$$\int_0^t e^{it'[3\xi\xi_1(\xi_1 - \xi) + t'[\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)]]} dt' = \frac{e^{it3\xi\xi_1(\xi_1 - \xi) + t[\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)]} - 1}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)}. \quad (4.7)$$

Now, inserting (4.7) in (4.6), one obtains

$$\widehat{f(t)}(\xi) = i\xi e^{it\xi^3} \int_{\mathbb{R}} \widehat{v}_0(\xi - \xi_1) \widehat{v}_0(\xi_1) \frac{e^{it3\xi\xi_1(\xi_1 - \xi) + t\Phi(\xi_1) + t\Phi(\xi - \xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)} d\xi_1. \quad (4.8)$$

$$\|f\|_{H^s}^2 \gtrsim \int_{-\gamma/2}^{\gamma/2} \langle \xi \rangle^{2s} \frac{\xi^2}{N^{4s}\gamma^2} \left| \int_{K_\epsilon} \frac{e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)} d\xi_1 \right|^2 d\xi, \quad (4.9)$$

where

$$K_\epsilon = \{\xi_1; \quad \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1; \quad \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}.$$

We have that $|K_\epsilon| \geq \gamma$ and

$$|3\xi\xi_1(\xi_1 - \xi)| \approx N^2\gamma. \quad (4.10)$$

In order to estimate (4.9) we consider two cases:

Case 1: $\xi - \xi_1 \in I_N, \xi_1 \in -I_N$. In this case

$$\begin{aligned} |\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| &= | -(-\xi_1)^p + |\xi|^p - (\xi - \xi_1)^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1) | \\ &\leq | -2(-1)^p \xi_1^p | + |(\xi^p - p\xi^{p-1}\xi_1 + \dots + p(-1)^{p-1}\xi\xi_1^{p-1}) + |\xi|^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1)|. \end{aligned} \quad (4.11)$$

Therefore,

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \leq C(N^p + N^{p-1}) \leq 2CN^p. \quad (4.12)$$

Case 2: $\xi_1 \in I_N, \xi - \xi_1 \in -I_N$. In this case

$$\begin{aligned} |\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| &= | -\xi_1^p + |\xi|^p - (-1)^p(\xi - \xi_1)^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1) | \\ &\leq | -2\xi_1^p | + |(-1)^p(\xi^p - p\xi^{p-1}\xi_1 + \dots + p(-1)^{p-1}\xi\xi_1^{p-1}) + |\xi|^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1)|. \end{aligned} \quad (4.13)$$

Therefore,

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \leq C(N^p + N^{p-1}) \leq 2CN^p. \quad (4.14)$$

From (4.10), (4.12) and (4.14) we conclude that for any $\xi_1 \in K_\epsilon$, we have

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)| \lesssim N^2(N^{p-2} + \gamma). \quad (4.15)$$

Similarly for any $\xi_1 \in K_\epsilon$, we get, for large N

$$\begin{aligned} \Phi(\xi_1) + \Phi(\xi - \xi_1) &= -|\xi_1|^p - |\xi - \xi_1|^p + \Phi_1(\xi_1) + \Phi_1(\xi - \xi_1) \leq -2N^p + CN^{p-1} \\ &\leq -N^p. \end{aligned} \quad (4.16)$$

Hence for $\gamma \ll 1$, one can obtain

$$\begin{aligned} \operatorname{Re} \left\{ e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)} \right\} &\leq e^{t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{-t\gamma^p/2} \\ &\leq e^{-tN^p} - e^{-t\gamma^p/2} \\ &\leq \frac{-e^{-t\gamma^p/2}}{2}. \end{aligned} \quad (4.17)$$

Combining (4.9), (4.15) and (4.17), using that $|z| \geq -\operatorname{Re} z$, we arrive

$$\|f\|_{H^s}^2 \gtrsim \gamma^{-2} N^{-4s} \gamma \langle \gamma \rangle^{2s} \gamma^2 \frac{(e^{-t\gamma^p})/2}{N^4(N^{p-2} + \gamma)^2} \gamma. \quad (4.18)$$

Taking $\gamma \sim 1$ and N very large, we obtain

$$\|f\|_{H^s}^2 \gtrsim N^{-4s-2p},$$

and this is a contradiction if $-4s - 2p > 0$ or equivalently $s < -\frac{p}{2}$. \square

Proof of Theorem 1.3. For $v_0 \in H^s(\mathbb{R})$, consider the Cauchy problem

$$\begin{cases} v_t + v_{xxx} + \eta Lv + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = \epsilon v_0(x), \end{cases} \quad (4.19)$$

where $\epsilon > 0$ is a parameter. The solution $v^\epsilon(x, t)$ of (4.19) depends on the parameter ϵ .

We can write (4.19) in the equivalent integral equation form as

$$v^\epsilon(t) = \epsilon V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t')dt', \quad (4.20)$$

where, $V(t)$ is the unitary group describing the solution of the linear part of the IVP (4.19).

Differentiating $v^\epsilon(x, t)$ in (4.20) with respect ϵ and evaluating at $\epsilon = 0$ we get

$$\left. \frac{\partial v^\epsilon(x, t)}{\partial \epsilon} \right|_{\epsilon=0} = V(t)v_0(x) =: v_1(x) \quad (4.21)$$

and

$$\frac{\partial^2 v^\epsilon(x, t)}{\partial \epsilon^2} \Big|_{\epsilon=0} = - \int_0^t V(t-t') \partial_x (v_1^2(x, t')) dt' =: v_2(x). \quad (4.22)$$

If the flow-map is C^2 at the origin from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$, we must have

$$\|v_2\|_{L_T^\infty H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R})}^2. \quad (4.23)$$

But from Proposition 4.1 we have seen that the estimate (4.23) fails to hold for $s < -\frac{p}{2}$ if we consider v_0 given by (4.4) and this completes the proof of the Theorem. \square

Now, we move to prove an ill-posedness results to the IVP (1.2)

Proposition 4.2. *Let $s < 1 - \frac{p}{2}$ and $T > 0$. Then there does not exist a space Y_T^s continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that*

$$\|V(t)u_0\|_{Y_T^s} \lesssim \|u_0\|_{H^s}, \quad (4.24)$$

$$\left\| \int_0^t V(t-t') (u_x(t'))^2 dt' \right\|_{Y_T^s} \lesssim \|u\|_{Y_T^s}^2, \quad (4.25)$$

Proof. Analogously as in the proof of Proposition 4.1 we consider the same v_0 as defined in (4.4), we take $u_0 := v_0$ and we calculate the H^s norm of $g(x, t)$, where

$$g(x, t) := \int_0^t V(t-t') [\partial_x V(t') u_0]^2 dt'. \quad (4.26)$$

We have

$$\widehat{g(t)}(\xi) = i\xi e^{it\xi^3} \int_{\mathbb{R}} (\xi - \xi_1) \widehat{v_0}(\xi - \xi_1) \xi_1 \widehat{u_0}(\xi_1) \frac{e^{it3\xi\xi_1(\xi_1 - \xi) + t\Phi(\xi_1) + t\Phi(\xi - \xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)} d\xi_1.$$

and

$$\|g\|_{H^s}^2 \gtrsim \int_{-\gamma/2}^{\gamma/2} \langle \xi \rangle^{2s} \frac{\xi^2}{N^{4s}\gamma^2} \left| \int_{K_\epsilon} \frac{\xi_1(\xi - \xi_1) e^{3it\xi\xi_1(\xi_1 - \xi) + t\Phi(\xi_1) + t\Phi(\xi - \xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)} d\xi_1 \right|^2 d\xi, \quad (4.27)$$

where

$$K_\epsilon = \{\xi_1; \quad \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1; \quad \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}.$$

Same way as in the proof of Proposition 4.1, we obtain

$$\|g\|_{H^s}^2 \gtrsim \gamma^{-2} N^{-4s} \gamma \langle \gamma \rangle^{2s} \gamma^2 \frac{N^4 e^{-t\gamma^p}}{(N^p + \gamma N^2)^2} \gamma. \quad (4.28)$$

Taking $\gamma \sim 1$ and N very large, we obtain

$$\|g\|_{H^s}^2 \gtrsim N^{-4s-2p+4},$$

and this is a contradiction if $-4s - 2p + 4 > 0$ or equivalently $s < 1 - \frac{p}{2}$. \square

Proof of Theorem 1.4. For $v_0 \in H^s(\mathbb{R})$, consider the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \eta Lu + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = \epsilon u_0(x), \end{cases} \quad (4.29)$$

where $\epsilon > 0$ is a parameter. The solution $u^\epsilon(x, t)$ of (4.29) depends on the parameter ϵ .

We can write (4.29) in the equivalent integral equation form as

$$u^\epsilon(t) = \epsilon V(t) u_0 - \int_0^t V(t-t') (u_x)^2(t') dt', \quad (4.30)$$

where, $V(t)$ is the unitary group describing the solution of the linear part of the IVP (4.29).

Differentiating $u^\epsilon(x, t)$ in (4.30) with respect ϵ and evaluating at $\epsilon = 0$ we get

$$\left. \frac{\partial u^\epsilon(x, t)}{\partial \epsilon} \right|_{\epsilon=0} = V(t) u_0(x) =: u_1(x) \quad (4.31)$$

and

$$\left. \frac{\partial^2 u^\epsilon(x, t)}{\partial \epsilon^2} \right|_{\epsilon=0} = - \int_0^t V(t-t') \partial_x (u_1^2(x, t')) dt' =: u_2(x). \quad (4.32)$$

If the flow-map is C^2 at the origin from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$, we must have

$$\|u_2\|_{L_T^\infty H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})}^2. \quad (4.33)$$

But from Proposition 4.2 we have that the estimate (4.33) fails to hold for $s < 1 - \frac{p}{2}$ if we consider $u_0 := v_0$ given by (4.4) and this completes the proof of the Theorem. \square

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